

LARGE SETS IN COUNTABLE AMENABLE GROUPS AND ITS APPLICATION

DIBYENDU DE AND PINTU DEBNATH

ABSTRACT. In the present paper our main objective is to extend the notion of D -sets in countable amenable groups and as a consequence we prove that a D^* -set in $(\mathbb{N}, +)$ is a D -set in (\mathbb{N}, \cdot) . Further we also discuss its connection with ergodicity and weak mixing for amenable group actions. We present an example of an amenable group, namely the group $(F_q[X], +)$, polynomial ring generated by finite field F_q , where IP^* -sets, D^* -sets and Central*-sets are all equivalent.

1. INTRODUCTION

Central sets in $(\mathbb{N}, +)$ were introduced by Furstenberg [11] and are known to have substantial combinatorial structure. For example, any Central set contains arbitrarily long arithmetic progressions, all possible finite sums of an infinite sequence, and solutions to all partition regular systems of homogeneous linear equations. In succession several notions of sets were introduced all of which have rich combinatorial structure like Central sets, for example Quasi Central sets [6], C-sets [7]. Another inclusion in this list is D -set [3]. Furstenberg's original definition of Central sets was in terms of notions of topological dynamics. Later Bergelson and Hindman [3] defined the notion of a Central set in an arbitrary semigroup $(S, +)$ in terms of the algebra of βS , the Stone-Ćecho compactification of S . They also defined notion of dynamical Central set, using the natural extension of Furstenberg's definition, and pointed out that any dynamical Central sets in $(S, +)$ is Central in $(S, +)$. Moreover, a result of Weiss (see [4, Theorem 6.11]) guarantees that in a countable semigroup $(S, +)$, a subset of S is Central if and only if it is dynamically Central. This equivalence of dynamically central and Central sets for arbitrary semigroup were finally proved in [18].

On the contrary to Central sets the notion of Quasi Central and C-sets were introduced using algebraic structure of βS . Equivalent dynamical characterization were established in [6] and [16] respectively.

The notion of D -set was introduced by Bergelson and Downarowicz in [3] for subsets of $(\mathbb{N}, +)$ using algebraic structure of $\beta\mathbb{N}$, and proved to be useful in connection with weak mixing for \mathbb{Z} -action. Authors also described dynamical characterization of D -sets in $(\mathbb{N}, +)$. Using dynamical characterization it can be easily proved that central sets are quasi central and quasi central sets are D -sets.

Date: September 30, 2014.

2000 Mathematics Subject Classification. Primary 05D10, Secondary 54H20.

Key words and phrases. Idempotents, IP -sets, Central Sets, Sets with Positive upper Banach density, Ergodicity, Weak Mixing.

The first author was partially supported DST PURSE Grant.

This work is a part of second authors Doctoral Dissertation and supported by CSIR grant.

To describe the Central, Quasi Central and D-sets, we shall need to pause and briefly introduce the algebraic structure of the Stone-Ćech compactification S of a discrete semigroup $(S, +)$. Given any discrete semigroup S we take βS to be the set of ultrafilters on S , identifying the points of S with the principal ultrafilters. The topology on βS has a basis consisting of $\{clA : A \subset S\}$, where $clA = \{p \in \beta S : A \in p\}$. The operation on S extends to βS , making S a left topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\lambda_p(q) = p + q$ from βS to βS is continuous. And, for each $x \in S$, the function $\rho_x : \beta S \rightarrow \beta S$ defined by $\rho_x(q) = q + x$ is continuous. (In spite of the fact that we are denoting the extension by $+$, the operation on S is very unlikely to be commutative. The center of $(\beta S, +)$ is equal to the center of S . Given p and q in βS and $A \subset S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in p\} \in q$).

Any compact left topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals and is also the union of all of the minimal right ideals. The intersection of any minimal left ideal with any minimal right ideal is a group and therefore contains idempotents. In particular, there are idempotents in $K(T)$. Such idempotents are called minimal. A subset L of T is a minimal right ideal if and only if $L = p + T$ for some minimal idempotent p . See [2] and [1] for an elementary introduction to the algebra of S and for any unfamiliar details. For any discrete semigroup S , a set $A \subset S$ is an IP-set if and only if A is a member of an idempotent in βS . A is an IP* set if and only if it is a member of every idempotent in βS . A subset $C \subset S$ is called central if it belongs to some idempotents of $K(\beta S)$. Like IP*-set a subset $C \subset S$ is called Central*-set if it belongs to every minimal idempotent of βS .

The notion of D -sets first introduced by Bergelson and Downarowicz in [3] for $(\mathbb{N}, +)$. It was defined analogously to Central -sets by replacing minimal idempotents by a wider class of idempotents all of whose members have positive upper Banach density, so that the class \mathcal{D} of D - sets is (strictly) intermediate between IP and Central sets. They also obtained a characterization of D -sets, analogous to that of IP-sets and Central-sets. To introduce the notion of D-sets let us recall the notion of upper Banach density.

Definition 1.1. A subset $A \subset \mathbb{Z}$ is said to have positive upper Banach density if

$$\bar{d}(A) = \limsup_{(m-n) \rightarrow \infty} \frac{|A \cap [n, m-1]|}{m-n} > 0.$$

An idempotent p in $\beta\mathbb{Z}$ is called *essential idempotent* if every member of p has positive upper Banach density. members of *essential idempotents* are called D -sets. Since the notion of essential idempotents not only depends on the algebraic structure of the Stone-Ćecho compactification, it's not straight forward to generalize this notion for arbitrary semigroup like minimal idempotents.

The paper is organized as follows. In section 2 we will introduce the notion of essential idempotents for countable amenable group G and then define the notion of D -sets in terms of topological dynamics. Finally it will be proved that a subset of an amenable group G is a D -set if and only if it belongs to some essential idempotents of βG .

Like IP* and Central* set, a subset of G will be called D^* -set if it belongs to every essential idempotent of βG . We denote the collection of all D^* -sets by \mathcal{D}^* and the union $\bigcup_{g \in G} (g\mathcal{D}^*)$ by denote by \mathcal{D}_+^* . In section 3 it will be shown that the

familiar ergodic-theoretic notions of ergodicity, weak mixing can be characterized by \mathcal{D}^* and \mathcal{D}_+^* sets.

Finally in section 4 we will discuss some results on Goldbach conjecture on \mathbb{N} and polynomial rings over finite fields. See [14] for an elementary introduction to polynomial rings over finite fields.

2. NOTION OF D - SETS FOR COUNTABLE AMENABLE GROUP

The notion of upper Banach density has a natural generalization for countable amenable groups. For this purpose let us first recall the notion of discrete amenable group.

Definition 2.1. A discrete group G is said to be amenable if there exists an invariant mean on the space $B(G)$ of real-valued bounded functions on G , that is, a positive linear functional $L : B(G) \rightarrow \mathbb{R}$ satisfying

1. $L(1_G) = 1$,
2. $L(f_g) = L(gf) = L(f)$ for all $f \in B(G)$ and $g \in G$, where $f_g(t) = f(tg)$ and $gf(t) = f(gt)$.

The existence of an invariant mean is only one item from a long list of equivalent properties. We will find the following characterization of amenability for discrete groups, which was established by Følner in [10], to be especially useful.

Theorem 2.2. A countable group G is amenable if and only if it has a left Følner sequence, namely a sequence of finite sets $F_n \subset G$, $n \in \mathbb{N}$, with $|F_n| \rightarrow \infty$ and such that

$$\frac{|F_n \cap gF_n|}{|F_n|} \rightarrow 1, \text{ for all } g \in G.$$

Proof. [17, Corollary 5.3]. □

Definition 2.3. Let G be a countable amenable group. A subset E of G is said to have positive Følner density or upper density with respect to some Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ provided that

$$\limsup_{n \rightarrow \infty} \frac{|E \cap F_n|}{|F_n|} > 0.$$

This will be denoted by $\bar{d}_{F_n}(E)$.

Definition 2.4. Let G be a countable amenable group. A subset E of G is said to have positive upper density if

$$\bar{d}(E) = \sup\{\bar{d}_{F_n}(E) : F_n \text{ is a Følner sequence in } G\} > 0.$$

From [10, Theorem 3] one can show that a set $A \subseteq G$ has positive upper density with respect to some Følner sequence if and only if there exists an invariant mean L on $B(G)$ such that $L(1_A) > 0$.

Definition 2.5. By a dynamical system we mean a pair $(X, \langle T_g \rangle_{g \in G})$ where

- (1) G is a semigroup
- (2) $\langle T_g \rangle_{g \in G}$ is continuous for all $g \in G$.
- (3) $g, h \in G$, we have $T_g \circ T_h = T_{gh}$.

Definition 2.6. A point y contained in the dynamical system $(X, \langle T_g \rangle_{g \in G})$ is said to be essentially recurrent if the set of visits $\{g \in G : T_g y \in U_y\}$ for any neighborhood U_y of y has positive upper density.

This can be easily observe that in any G be a countable amenable group syndetic sets always have positive upper density. This shows that every uniformly recurrent point is essentially recurrent. A characterization of essentially recurrent points in terms of the properties of their orbit closures is provided below.

Definition 2.7. A dynamical system $(Y, \langle T_g \rangle_{g \in G})$ will be called measure saturated if every nonempty open set U there exists an invariant measure μ such that $\mu(U) > 0$.

Theorem 2.8. A point y in a dynamical system $(Y, \langle T_g \rangle_{g \in G})$ is essentially recurrent if and only if the orbit closure $\overline{O}(y) = \overline{\{T_g(y) : g \in G\}}$ is measure saturated.

Proof. First let us show that if a point y in a dynamical system $(Y, \langle T_g \rangle_{g \in G})$ is essentially recurrent then the orbit closure $\overline{O}(y) = \overline{\{T_g(y) : g \in G\}}$ is measure saturated. Let U_y be an open neighborhood of y containing a closed neighborhood U . Since y is essentially recurrent, the set $A = \{g \in G : T_g y \in U_y\}$ has positive upper density d with respect to a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ in G . Then

$$\limsup_{n \rightarrow \infty} \frac{|E \cap F_n|}{|F_n|} > 0.$$

Passing to a subsequence we can say that there exists a sequence $\{F_n\}_{n \in \mathbb{N}}$ in G with $|F_n| \rightarrow \infty$, such that $\frac{|E \cap F_n|}{|F_n|}$ converges to d . Let $\{\mu_n : n \in \mathbb{N}\}$ be the normalized counting measures supported by the sets $\{T_g(y) : g \in F_n\}$. Then it is easy to observe that for each $n \in \mathbb{N}$ we have $\mu_n \circ T_g = \mu_n$. Let μ be a weak limit of the sequence $(\mu_n)_{n \in \mathbb{N}}$, where a sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ converges to μ weak if $\int f \mu_n \rightarrow \int f \mu$ for every continuous function f on the space Y . Clearly μ is also T_g invariant for each $g \in G$ and supported by $\overline{O}(y) = \{T_g(y) : y \in G\}$ and satisfies $\mu(U) > 0$, and thus $\mu(U_y) > 0$.

Now for any invariant measure μ carried by $\overline{O}(y)$ we define

$$M_\mu = \{x \in \overline{O}(y) : \text{there exists a neighborhood } V \text{ of } x \text{ such that } \mu(V) > 0\}.$$

Let M be the closure of union of all M_μ 's. Then $y \in M$ and since M is a closed invariant set, it follows that $M = \overline{O}(y)$, i.e., $\overline{O}(y)$ is measure saturated.

Conversely, assume that $\overline{O}(y)$ is measure saturated. Let $U_y \ni y$ be an open set. Then there exist an invariant measure μ supported by $\overline{O}(y)$ such that $\mu(U_y) = a > 0$. We have to show that the set $\{g \in G : T_g y \in U_y\}$ has positive upper density. Since G is a countable amenable group there exists a Følner sequence say $(F_n)_{n \in \mathbb{N}}$, namely a sequence of finite sets $F_n \subset G$, $n \in \mathbb{N}$ with $|F_n| \rightarrow \infty$ and such that $\frac{|F_n \cap g F_n|}{|F_n|} \rightarrow 1$ for all $g \in G$. Let us now set

$$f_n(x) = \frac{1}{|F_n|} \sum_{g \in F_n} 1_{U_y}(T_g(x)).$$

Then we have that $0 \leq f_n(x) \leq 1$ for all x and since T_g 's are measure preserving we have that $\int f_n d\mu \geq a > 0$ for all $n \in \mathbb{N}$. Let $f(x) = \limsup_{n \rightarrow \infty} f_n(x)$. By

Fatso's Lemma, we have

$$\int f d\mu = \int \limsup_{n \rightarrow \infty} f_n(x) d\mu \geq \limsup_{n \rightarrow \infty} \int f_n(x) \geq a.$$

Since μ is supported on $\overline{O}(y)$ there exists $y' \in \overline{O}(y)$ such that $f(y') = a > 0$. Now let us set $R = \{g \in G : T_g y' \in U_y\}$. Since

$$f(y') = \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{g \in F_n} 1_{U_y}(T_g(x)),$$

it follows that R has Følner density a . Now for any $g \in G$ with $T_g y' \in U_y$ there exists a neighborhood V of y' such that $T_g(V) \subset U_y$. Choose $g' \in G$ such that $T_{gg'} y \in V$ so that $T_{gg'} y \in U_y$. It follows that $\{g \in G : T_g y \in U_y\}$ has positive upper density and hence y is essentially recurrent point. \square

Let us extend the notion of essential idempotent in $\beta\mathbb{Z}$ for countable amenable group.

Definition 2.9. Let G be a countable amenable group. An idempotent $p \in \beta G$ is said to be an essential idempotent if every member of p has positive upper density.

For any $x \in G$, let us set $r_x : G \rightarrow G$ by $r_x(g) = gx$. Now βG becomes a compact left topological semigroup under the natural extension $\lambda_q(p) = q \cdot p$. Let us first characterize essentially recurrent points in the dynamical system $(\beta G, \langle \lambda_q \rangle_{q \in \beta G})$.

Lemma 2.10. *Let G be a countable amenable group. An idempotent q in βG is an essentially recurrent point in the topological dynamical system $(\beta G, \langle \lambda_q \rangle_{q \in \beta G})$, if and only if it is essential idempotent in $(\beta G, \cdot)$.*

Proof. Let q be essentially recurrent point in the topological dynamical system $(\beta G, \langle \lambda_q \rangle_{q \in \beta G})$ and let E be any element of q . We have to show that E has positive upper density in the amenable group G . The closure \overline{E} of E in βG can be interpreted as a neighborhood of q . Since q is essentially recurrent, $\overline{O}(q)$ is measure saturated and hence exists an invariant measure μ such that $\mu(\overline{E}) > 0$. Choose $g \in G$ such that $g \in \overline{E}$, $g \in E$. Since μ is supported by the orbit closure of the identity the set $\{g \in G : \lambda_g(e) \in \overline{E}\}$ has positive upper density $(\beta G, \langle \lambda_p \rangle_{p \in \beta G})$. But the $\{g \in G : \lambda_g(e) \in \overline{E}\} = E$. This implies that E has positive upper density, and hence q is an essential idempotent in $(\beta G, \cdot)$.

To prove the converse consider the map defined by $\lambda_q(p) = q \cdot p$ onto $\overline{O}(q)$ and both e and q map to q . A neighborhood U_q of q in $\overline{O}(q)$ lifts to a neighborhood V_q of q in βG and the set $R_q = \{g \in G : \lambda_g(q) \in U_q\}$ contains the set $R_e = \{g \in G : \lambda_g(e) \in V_q\}$ in fact $g \in V_q$ implies that $\lambda_g(q) \in U_q$ as $qU_q = V_q$. But the set R_e is a member of q (because its complement is not). Since q is assumed to be an essential idempotent, all members of q have positive upper density. It follows that R_e has positive upper density and hence, q is essentially recurrent. \square

To prove our main theorem let us recall the following lemma from [4].

Lemma 2.11. *Let $\pi : X \rightarrow Y$ be a topological factor map (surjection) between dynamical systems (X, S) and (Y, T) . If y is an essentially recurrent point in Y then there exists an essentially recurrent π -lift x of y . Moreover, we can find such x for which $\overline{O}(x)$ contains no proper closed invariant subset which is mapped by π onto $\overline{O}(y)$.*

Now we are in a position to prove our main Theorem of this section.

Theorem 2.12. *Let G be a countable amenable group. A set $D \subset G$ is a D -set if and only there exists a compact dynamical system $(X, \langle T_g \rangle_{g \in G})$, points $x, y \in X$ with x, y proximal and y essentially recurrent and an open neighborhood U_y of y such that*

$$D = \{g \in G : T_g(x) \in U_y\}.$$

Proof. Let $D = \{g \in G : T_g(x) \in U_y\}$ where x, y and U_y , are as in the formulation of the theorem. Since x, y are proximal $p\text{-}\lim_{g \in G} T_g x = p\text{-}\lim_{g \in G} T_g y$. Consider a factor map $\pi : \beta G \rightarrow \overline{O}(y)$ defined by $p \rightarrow p\text{-}\lim_{g \in G} T_g y$. By previous lemma we can find in βG an essentially recurrent point p_1 which is π -lift of y , and whose orbit closure is a minimal lift of $\overline{O}(y)$. We will show that p_1 can be replaced by an idempotent. Consider the set

$$I = \{p \in \overline{O}(p_1) : \pi(p) = y\}.$$

It can be easily verified that I is a closed sub semigroup of βG , so it contains an idempotent q . Since $\pi(p) = y$, its orbit closure maps onto $\overline{O}(y)$. By minimality of the lift $\overline{O}(p_1)$, q has the same orbit closure as p_1 , and hence is essentially recurrent.

Since $\pi(q) = q\text{-}\lim_{g \in G} T_g y = p\text{-}\lim_{g \in G} T_g x$ and U_y is a neighborhood of y we have that $\{g \in G : T_g x \in U_y\} \in q$ and therefore $D \in q$.

Conversely we consider the dynamical system $(X, \langle T_g \rangle_{g \in G})$, where $X = \{0, 1\}^G$ and for each $g \in G$, $T_g : X \rightarrow X$ defined by $T_g(f) = f \circ \lambda_g$. Let D be a D -set. Then there exists an essential idempotent $q \in G$ such that $D \in q$. Consider the characteristic function $x = 1_D \in X$. Let $y = q\text{-}\lim T_g x$. Since q is an idempotent $y = q\text{-}\lim T_g y = q\text{-}\lim T_g(q\text{-}\lim T_g x)$. Since y is the image of q via the factor map $\pi : \beta G \rightarrow \overline{O}(y)$ given by $p \rightarrow p\text{-}\lim T_g y$ and q is essentially recurrent point in $(\beta G, \langle \delta_x \rangle_{x \in G})$, we have that q is essentially recurrent point.

Now let $U_y = \{z \in X : z(e) = y(e)\}$ is a neighborhood of y in X . Then U_y is a neighborhood of y in X . We note that $y(e) = 1$. In fact $y = q\text{-}\lim T_g(x)$ so that $\{g \in G : T_g x \in U\} \in q$. This implies that $\{g \in G : T_g x \in U_y\} \cap D \neq \emptyset$. Let us choose $g \in D$ such that $T_g(x) \in U_y$. Then $y(e) = T_g(x)(e) = x(eg) = 1$. Thus given any $g \in G$,

$$\begin{aligned} g \in D &\Leftrightarrow x(g) = 1 \\ &\Leftrightarrow T_g(x)(e) = 1 \\ &\Leftrightarrow T_g(x) \in U_y. \end{aligned}$$

□

This theorem implies that every Central set in an amenable group is a D -set. We also have the following combined additive and multiplicative property.

Theorem 2.13. *Every D^* -set in $(\mathbb{N}, +)$ is a D -set in (\mathbb{N}, \cdot) .*

3. MIXING PROPERTIES

In this section we will discuss now the connections between essential idempotents and unitary actions of countable amenable groups. We shall consider a unitary representation $(U_g)_{g \in G}$ on the Hilbert space $L^2(X, \mathcal{B}, \mu)$ defined by $U_g f(x) = f(T_g x)$.

Definition 3.1. Let $(U_g)_{g \in G}$ be a unitary representation of a group G on a separable Hilbert space H .

- (i) A vector $x \in H$ is called compact if the set $\{U_g x : g \in G\}$ is totally bounded in H .
- (ii) The representation $(U_g)_{g \in G}$ is called weak mixing if there are no nonzero compact vectors.

Theorem 3.2. *Given a unitary representation $(U_g)_{g \in G}$ of a group G on a separable Hilbert space H let*

$$H_c = \{x \in H : x \text{ is compact with respect to } (U_g)_{g \in G}\}.$$

Then the restriction $(U_g)_{g \in G}$ to the invariant subspace $H_{wm} = H_c^\perp$ is weak mixing.

Recall that in a Hilbert space the norm convergence $\lim x_n = y$ is equivalent to the conjunction of the weak convergence of x_n to y and the convergence of norms $\lim \|x_n\| = \|y\|$. Since any unitary operator U is an isometry, the relation $p\text{-}\lim U_g x = x$ for some $p \in \beta G$ holds in the weak topology if and only if it holds in the strong topology. The following lemma follows from [2, Theorem 4.3].

Lemma 3.3. *If $p \in \beta G$ is an idempotent then for any $x \in H_c$ one has $p\text{-}\lim U_g x = x$.*

The above statement can be reversed for essential idempotents.

Lemma 3.4. *If $p \in \beta G$ is an essential idempotent and $p\text{-}\lim U_g x = x$ for some $x \in H$ then $x \in H_c$.*

Proof. For $\epsilon > 0$ consider the set $E = \{g \in G : \|U_g x - x\| < \epsilon/2\}$. Since $p\text{-}\lim U_g x = x$, we have $E \in p$. Note that for any $g_1, g_2 \in E$ one has

$$\|T_{g_1 g_2^{-1}} x - x\| = \|T_{g_1} x - T_{g_2} x\| \leq \|T_{g_1} x - x\| + \|T_{g_2} x - x\| \leq \epsilon.$$

Since E has positive upper Banach density, this implies that EE^{-1} is syndetic. Since U is an isometry we have covered the orbit of x by finitely many ϵ -balls, hence the orbit of x is precontract, so that $x \in H_c$. \square

Lemma 3.5. *If $p \in \beta G$ is an essential idempotent then for any $x \in H_{wm}$ one has $p\text{-}\lim U_g x = 0$ weak.*

Proof. By compactness of the ball of radius $\|x\|$ around zero in the weak topology, there exists some y such that $p\text{-}\lim U_g x = y$ weak. Since H_{wm} is closed and invariant under U , we have $y \in H_{wm}$. On the other hand, p is an idempotent, by Lemma 3.3 $p\text{-}\lim U_g y = y$. By Lemma 3.4, $y \in H_c$. This implies $y = 0$. \square

One can show that unitary representation $(U_g)_{g \in G}$ is weak mixing if and only if in the decomposition $H = H_c \oplus H_{wm}$ one has $H_c = \{0\}$. Let now $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ be an invertible weakly mixing system. It can be easily checked that in this case the unitary operator induced by $\langle T_g \rangle_{g \in G}$ on $L^2(\mu)$ is weakly mixing in the above sense on the orthocomplement of the space of constant functions. This allows us to work with the unitary operator $(U_g)_{g \in G}$ acting on the Hilbert space H as $L_0^2(X, \mathcal{B}, \mu) = \{f \in L^2(X, \mathcal{B}, \mu) : \int f d\mu = 0\}$.

Remark 3.6. It follows from the previous discussions that an invertible probability measure preserving system $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weakly mixing if and only if for any $A, B \in \mathcal{B}$ and any essentially recurrent idempotent p , $p\text{-}\lim_g \mu(A \cap T_g B) = \mu(A) \mu(B)$.

Let us recall here deep Theorem of Lindenstrauss for our purpose.

Definition 3.7. A sequence of sets F_n will be said to be tempered if for some $C > 0$ and all n , $|\bigcup_{k \leq n} F_k^{-1} F_{n+1}| \leq C |F_{n+1}|$.

Theorem 3.8 (Lindenstrauss Pointwise ergodic theorem). *Let G be an amenable group acting ergodically on a measure space (X, \mathcal{B}, μ) , and let F_n be a tempered Følner sequence. Then for any $f \in L^1(\mu)$,*

$$\lim_n \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = \int f(x) d\mu(x) \text{ a. e.}$$

Theorem 3.9. *An invertible measure preserving system $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is ergodic if and only if for any $A, B \in \mathcal{B}$ and $\epsilon > 0$ the set $R_{A,B}^\epsilon = \{g \in G : \mu(A \cap T_g B) > \mu(A)\mu(B) - \epsilon\}$ belongs to \mathcal{D}_+^* .*

To prove the above Theorem we need the following Lemma. From now to express “measure preserving system” we will also write m.p.s.

Lemma 3.10. *Any invertible probability m.p.s. $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is ergodic if and only if $N(A, B) = \{g \in G : \mu(T_g A \cap B) > 0\} \neq \emptyset$ for any $A, B \in \mathcal{B}$ such that $\mu(A)\mu(B) > 0$.*

Proof. Let $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ be a ergodic measure preserving system. Then for any tempered Følner sequence $\langle F_n \rangle_{n \in \mathbb{N}}$,

$$\lim_n \frac{1}{|F_n|} \sum_{g \in F_n} 1_A(T_g x) = \int 1_A(x) d\mu(x) = \mu(A) \text{ a.e.}$$

Then dominated convergence theorem we have

$$\lim_n \frac{1}{|F_n|} \sum_{g \in F_n} \int 1_A(T_g x) 1_B(x) d\mu(x) = \mu(A) \int 1_B d\mu = \mu(A)\mu(B) > 0.$$

i.e.

$$\lim_n \frac{1}{|F_n|} \sum_{g \in F_n} \mu(T_g A \cap B) = \mu(A)\mu(B) > 0.$$

Hence there exists $g_0 \in G$ such that $\mu(T_{g_0} A \cap B) > 0$. This implies, $N(A, B) \neq \emptyset$.

Conversely let $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is not ergodic. Then there exists $A \in \mathcal{B}$ with $0 < \mu(A) < 1$ such that $T_g A = A$ for all $g \in A$. This implies that $0 = \mu(A \cap A^c) = \mu(T_g A \cap A^c)$. This contradicts the fact that $N(A, B) \neq \emptyset$. \square

Proof of Theorem 3.9. Assume that $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is ergodic. Denote $f = 1_A$ and $h = 1_B$. Decompose $h = h_1 + h_2$, $h_1 \in H_c$, $h_2 \in H_{wm}$. Clearly $\int h_1 d\mu = \mu(B)$. Now let $\langle F_n \rangle_{n \in \mathbb{N}}$ be tempered Følner sequence. Then

$$\frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x) \rightarrow \int f(x) d\mu(x) = \mu(A) \text{ a.e.}$$

from dominated convergence theorem we have

$$\frac{1}{|F_n|} \sum_{g \in F_n} \int f(T_g x) g_1(x) d\mu(x) \rightarrow \mu(A) \int g_1 d\mu = \mu(A) \mu(B).$$

hence there exist g_o satisfying $\int f(T_{g_o} x) g_1(x) d\mu(x) > \mu(A) \mu(B) - \epsilon$. Let p be an essential idempotent. Applying Lemma 3.3 and 3.4, we can write

$$\begin{aligned} p\text{-lim} \mu(T_{g_o} A \cap T_g B) &= p\text{-lim} \int f(T_{g_o} x) g(T_g x) d\mu(x) \\ &= p\text{-lim} \int f(T_{g_o} x) g_1(T_g x) d\mu(x) + p\text{-lim} \int f(T_{g_o} x) g_2(T_g x) d\mu(x) \\ &= \int f(T_{g_o} x) g_1(x) d\mu(x) > \mu(A) \mu(B) - \epsilon. \end{aligned}$$

this implies that $g_o^{-1} R_{A,B}^\epsilon \in p$, which proves that $R_{A,B}^\epsilon$ is D_+^* set.

The converse implication is obvious : if the sets $R_{A,B}^\epsilon$ are D_+^* then they are nonempty which implies ergodicity. Previous Lemma. \square

Theorem 3.11. *The system $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weakly mixing if and only if for any $A, B \in \mathcal{B}$ and $\epsilon > 0$ the set $R_{A,B}^\epsilon$ is D^* .*

Proof. Assume that $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weakly mixing. Then by Remark 3.5 for any $A, B \in \mathcal{B}$ and any essential idempotent p , we have $p\text{-lim}_g \mu(A \cap T_g B) = \mu(A) \mu(B)$ and hence $R_{A,B}^\epsilon$ is a D^* set.

To prove the converse part let us assume that

$$L_{A,B}^\epsilon = \{g \in G : \mu(A \cap T_g B) < \mu(A) \mu(B) + \epsilon\}.$$

It proved that $L_{A,B}^\epsilon$ is also a D^* -set. Now $g \in R_{A^c,B}^\epsilon$ implies that $\mu(A^c \cap T_g B) > \mu(A^c) \mu(B) - \epsilon$. But

$$\mu(A^c \cap T_g B) + \mu(A \cap T_g B) = \mu(T_g B) = \mu(B).$$

So

$$g \in R_{A^c,B}^\epsilon \Rightarrow \mu(B) - \mu(A \cap T_g B) > \mu(A^c) \mu(B) - \epsilon \Rightarrow \mu(A \cap T_g B) < \mu(A) \mu(B) + \epsilon.$$

Hence $L_{A,B}^\epsilon$ is D^* -set.

Now $L_{A,B}^\epsilon \cap R_{A,B}^\epsilon = \{g \in G : |\mu(A \cap T_g B) - \mu(A) \mu(B)| < \epsilon\}$ is D^* -set. Thus for any essential idempotent p , we have $p\text{-lim}_g \mu(A \cap T_g B) = \mu(A) \mu(B)$. This implies $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weak mixing. \square

Remark 3.12. It is worth to note that if the system $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weak mixing then $R_{A,B}^\epsilon$ is of density one. In fact from [5] we know that for countable discrete amenable group G the system $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weak mixing iff for any Følner sequence $\{F_n\}$ and $A, B \in \mathcal{B}$ we have $\frac{1}{|F_n|} \sum_{g \in F_n} |\mu(A \cap T_g B) - \mu(A) \mu(B)| \rightarrow 0$ as

$n \rightarrow \infty$. From this it can be estimated that $R_{A,B}^\epsilon \cap L_{A,B}^\epsilon$ is of density one i.e $R_{A,B}^\epsilon$ is of density one. This can also be proved by using Lindenstrauss pointwise ergodic Theorem.

Theorem 3.13. *Let G countable discrete amenable group acting on a probability space (X, B, μ) . Then $(X, \mathcal{B}, \mu, \langle T_g \rangle_{g \in G})$ is weak mixing iff for any Følner sequence $\langle F_n \rangle$ and $A, B \in \mathcal{B}$, $R_{A,B}^\epsilon$ is of density one.*

Proof. Since T_g is weak mixing in particular ergodic for any tempered Følner sequence $\langle F_n \rangle$ in G

$$\frac{1}{|F_n|} \sum_{g \in F_n} 1_B(T_g x) \rightarrow \int 1_B(x) d\mu(x) = \mu(B) \text{ a.e.}$$

Now applying dominated convergence theorem, we have

$$\frac{1}{|F_n|} \sum_{g \in F_n} \int 1_B(T_g x) 1_A(x) d\mu(x) \rightarrow \mu(B) \int 1_A d\mu = \mu(A) \mu(B).$$

Hence

$$\frac{1}{|F_n|} \sum_{g \in F_n} \mu(A \cap T_g B) \rightarrow \mu(A) \mu(B).$$

Now using the equality $(A \cap T_g B) \times (A \cap T_g B) = (A \times A) \cap (T_g \times T_g)(B \times B)$ we have

$$\begin{aligned} & \frac{1}{|F_n|} \sum_{g \in F_n} (\mu \times \mu)((A \cap T_g B) \times (A \cap T_g B)) \\ &= \frac{1}{|F_n|} \sum_{g \in F_n} \mu \times \mu((A \times A) \cap T_g \times T_g(B \times B)) \rightarrow \mu \times \mu(A \times A) \mu \times \mu(B \times B) \end{aligned}$$

hence

$$\frac{1}{|F_n|} \sum_{g \in F_n} \mu(A \cap T_g B)^2 \rightarrow \mu(A)^2 \mu(B)^2$$

Thus

$$\begin{aligned} & \frac{1}{|F_n|} \sum_{g \in F_n} (\mu(A \cap T_g B) - \mu(A) \mu(B))^2 \\ &= \frac{1}{|F_n|} \sum_{g \in F_n} \mu(A \cap T_g B)^2 - 2\mu(A) \mu(B) \frac{1}{|F_n|} \sum_{g \in F_n} \mu(A \cap T_g B) + \mu(A)^2 \mu(B)^2 \rightarrow 0. \end{aligned}$$

Now if possible let $\frac{|S \cap F_n|}{|F_n|} \rightarrow l > 0$, where $S = G \setminus (R_{A,B}^\epsilon \cap L_{A,B}^\epsilon)$. Then

$$\begin{aligned} & \frac{1}{|F_n|} \sum_{g \in F_n} (\mu(A \cap T_g B) - \mu(A) \mu(B))^2 \\ & \geq \frac{1}{|F_n|} \sum_{g \in S \cap F_n} (\mu(A \cap T_g B) - \mu(A) \mu(B))^2 \\ & \geq \frac{1}{|F_n|} \sum_{g \in S \cap F_n} \epsilon^2 = \epsilon^2 \frac{|S \cap F_n|}{|F_n|} \rightarrow \epsilon^2 l > 0, \end{aligned}$$

a contradiction . □

4. FEW WORDS ON GOLDBACH CONJECTURE

If Goldbach conjecture is true then we have $2\mathbb{N} \subset P+P$. Author in [9] established that $2\mathbb{N} \setminus P+P$ is of natural density zero. This shows that if A be a Central-set in \mathbb{N} it intersects $2\mathbb{N}$ centrally i.e. in a set with positive upper density and therefore meets $P+P$. Hence $P+P$ becomes a Central*-set. But it is unknown to us, whether $P+P$ is an IP*-set.

In the following we provide an example of D*-sets in polynomial ring over finite fields F_q where $\text{Char } F_q \neq 2$. In case of $(\mathbb{Z}, +)$ we know that any ideal generated by an integer is an IP*-set. However in the following Theorem we shall extend this result for $(F_q[X_1, X_2, \dots, X_k], +)$ in its full generality.

Theorem 4.1. *The ideal $\langle f_1(X_1), \dots, f_k(X_k) \rangle$ generated by $f_1(X_1), \dots, f_k(X_k)$ in the polynomial ring $(F_q[X_1, X_2, \dots, X_k], +, \cdot)$ over finite field F_q is an IP*-set. in the group $(F_q[X_1, X_2, \dots, X_k], +)$.*

Proof. We will show that ideal $\langle f_1(X_1), f_2(X_2), \dots, f_k(X_k) \rangle$ is an IP*-set and hence a D*-set in $(F_q[X_1, X_2, \dots, X_k], +)$.

For simplicity we work with $k = 2$. Let $\langle g_n(X_1, X_2) \rangle_{n=1}^\infty$ be a sequence in $F_q[X_1, X_2]$.

Let $g(X_1, X_2)$ be a polynomial in $F_q[X_1, X_2]$. Then

$$g(X_1, X_2) = \sum_{i \leq n, j \leq m} a_{i,j} X_1^i X_2^j, \text{ where } a_{i,j} \in F_q.$$

Since $X_1^i, f_1(X_1) \in F_q[X_1]$ and $X_2^j, f_2(X_2) \in F_q[X_2]$ by applying division algorithm we have

$$X_1^i = f_1(X_1)q_i(X_1) + r_i(X_1), \text{ where } \deg(r_i(X_1)) < \deg f_1(X_1)$$

$$X_2^j = f_2(X_2)q_j(X_2) + r_j(X_2), \text{ where } \deg(r_j(X_2)) < \deg f_2(X_2).$$

Then $g(X_1, X_2)$ can be expressed as

$$\sum_{i \leq n, j \leq m} a_{i,j} (f_1(X_1)q_i(X_1) + r_i(X_1))(f_2(X_2)q_j(X_2) + r_j(X_2)).$$

$$g(X_1, X_2) = f_1(X_1)h_1(X_1, X_2) + f_2(X_2)h_2(X_1, X_2)$$

$$+ \sum_{\substack{\deg(r_i(X_1)) < \deg f_1(X_1) \\ \deg(r_j(X_2)) < \deg f_2(X_2)}} a_{i,j} r_i(X_1) r_j(X_2).$$

Therefore we can write

$$g(X_1, X_2) = h(X_1, X_2) + r(X_1, X_2),$$

where

$$h(X_1, X_2) \in \langle f_1(X_1), f_2(X_2) \rangle$$

and $r(X_1, X_2)$ is a polynomial such that $\deg r(X_1, X_2) < \deg f_1(X_1) + \deg f_2(X_2)$.

This implies that

$$g_n(X_1, X_2) = h_n(X_1, X_2) + r_n(X_1, X_2)$$

where

$$h_n(X_1, X_2) \in \langle f_1(X_1), f_2(X_2) \rangle$$

and $r_n(X_1, X_2)$ is a polynomial such that $\deg r_n(X_1, X_2) < \deg f_1(X_1) + \deg f_2(X_2)$.

But the set $\{r_n(X_1, X_2) : n \in \mathbb{N}\}$ is finite. Since $\{g_n(X_1, X_2) : n \in \mathbb{N}\}$ is infinite there exists p many polynomials $g_{n_i}(X_1, X_2) : i = 1, 2, \dots, p$ such that the corresponding $r_{n_i}(X_1, X_2)$ for $i = 1, 2, \dots, p$ are equals. Now adding we get

$$\sum_{i=1}^p g_{n_i}(X_1, X_2) = \sum_{i=1}^p h_{n_i}(X_1, X_2) + \sum_{i=1}^p r_{n_i}(X_1, X_2).$$

This implies that

$$\sum_{i=1}^p g_{n_i}(X_1, X_2) \in \langle f_1(X_1), f_2(X_2) \rangle$$

as

$$\sum_{i=1}^p r_{n_i}(X_1, X_2) = 0.$$

Therefore $\langle f_1(X_1), f_2(X_2) \rangle$ is an IP*-set and hence D*-set. \square

Let us state first the analogue Goldbach conjecture for the function field $F_q[X]$.

Goldbach conjecture for the polynomials over finite field : For a monic polynomial $m(X)$ with degree ≥ 2 , there exist two monic irreducible polynomial $f_1(X)$ and $f_2(X)$ with $\deg f_2(X) < \deg f_1(X) = \deg m(X)$ such that $m(X) = f_1(X) + f_2(X)$.

For any polynomial $m(X)$ there exists $\alpha \in F_q$ such that $\alpha m(X)$ is monic and if above conjecture is true then $\alpha m(X) = f_1(X) + f_2(X) \Rightarrow m(X) = \alpha^{-1} f_1(X) + \alpha^{-1} f_2(X)$. Hence this implies that

$$P_X + P_X = \{f_1(X) + f_2(X) : \text{irreducible polynomial } f_1(X) \text{ and } f_2(X)\}$$

is a Central*-set. But it is not known to us that whether like $P + P$, the following set $P_X + P_X$ is a Central*-set. We conclude this article with the following thm which shows that *notions are equivalent.

Theorem 4.2. *Any IP*-set containing 0 in $(F_q[X], +)$ contains an ideal of the form $\langle X^m \rangle$ for some $m \in \mathbb{N}$.*

Proof. Let us claim that any syndetic IP set A in $(F_q[X], +)$ contains $\langle X^m \rangle$ for some $m \in \mathbb{N}$. Now A being a syndetic set will be of the form

$$A = \bigcup_{i=1}^k (f_i(X) + \langle X^m \rangle) \text{ (modulo finite terms)}$$

for some $m, k \in \mathbb{N}$ with $m > \deg f_i(X)$. Again since A is an IP-set one of $f_i(X)$ must be zero.

We end this article with the following observation. \square

Remark 4.3. We know that for arbitrary countable discrete amenable group, $\text{IP}^* \Rightarrow \text{D}^* \Rightarrow \text{C}^*$. In $(\mathbb{N}, +)$ the converse implications do not hold. But from Theorem 4.1 and 4.2 it follows that converse implications do hold for $F_q[X]$.

REFERENCES

1. V. Bergelson, *Ergodic Ramsey theory - an update*, *Ergodic theory of \mathbb{Z}^d actions*, London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996., pp. 1- 61.
2. V. Bergelson, *Minimal idempotents and ergodic Ramsey Theory*, *Topics in dynamics and ergodic theory*, London. Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2003., vol 310, pp. 8-39.
3. V. Bergelson and T. Downarowicz, *Large Sets of Integers and Hierarchy of Mixing Properties of Measure Preserving Systems*, *Colloquium Mathematicum* 110 (2008), no. 1, 117-150.
4. V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and ergodic Ramsey theory*, *Trans. Amer. Math. Soc.* 320 (1990), 293-320.
5. V. Bergelson and J. Rosenblatt, *Mixing actions of groups*, *Illinois J. Math.* 32 (1) (1988), 65-80.
6. S. Burns and N. Hindman, *Quasi-central sets and their dynamical characterization*, *Topology Proceedings* 31 (2007), 445-455.
7. D. De, N. Hindman, D. Strauss, *A new and stronger central sets theorem*, *Fundamenta Mathematicae* 199 (2008), 155-175.
8. R. Ellis, *Distal transformation groups*, *Pacific J. Math.* 8 (1958), 401-405.
9. T. Estermann, *On the Goldbach's problem : Proof that almost all even positive integers are sums of two primes*, *Proc. London Math. Soc. Ser. 2* 44 (1938) 307-314.
10. E. Følner *On groups with full Banach mean values*, *Math. Scand.* 2 3 (1955) 243-254.
11. H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.
12. N. Hindman, A. Maleki, and D. Strauss, *Central sets and their combinatorial characterization*, *J. Comb. Theory (Series A)* 74 (1996), 188-208.
13. N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.
14. T. H. Le, *Green-Tao theorem in function fields*, *Acta Arith.* 147 (2011), no. 2, 129-152.
15. J. Li, *Dynamical characterization of C-sets and its application*, *Fundamenta Mathematicae* 199 (2008), 155-175.
16. E. Lindenstrauss, *Pointwise theorems for amenable groups*, *Invent. Math.* 146 (2001), 259-295.
17. I. Namioka, *Følner's condition for amenable semi-groups*, *Math. Scand.* 15 (1964), 18-28.
18. H. Shi and H. Yang, *Nonmetrizable topological dynamical characterization of central sets*, *Fund. Math.* 150 (1996), 1-9.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KALYANI, INDIA
E-mail address: dibyendude@klyuniv.ac.in

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KALYANI, INDIA
E-mail address: debnathpintu03@gmail.com